Recall from last time:

Theorem (Landau)

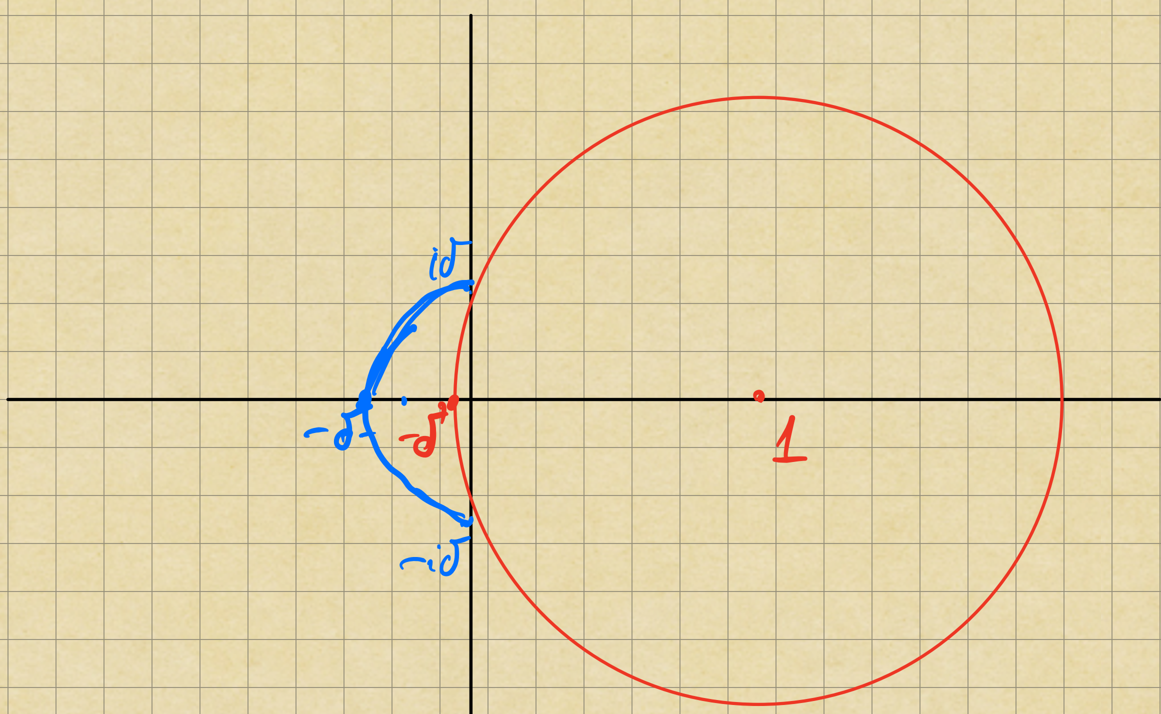
Let $f \in \mathcal{A}$ with $\sigma_c(f) < \infty$ and $f(n) \geq 0$, $\forall\, n \in \mathbb{N}$. Then $L_f(s)$ has singularity (i.e. not holomorphic) at $s = \sigma_c(f)$. (pole)

Proof:

By replacing $f(n)$ with $f(n) n^{-\sigma_c(f)}$, we may assume $\sigma_c(f) = 0$.

Suppose for contradiction $L_f(s)$ is analytic at $s=0$. Then it is holomorphic in a neighbourhood of $s=0$. We may assume $L_f(s)$ analytic in

$$D := \{s : \mathrm{Re}(s) > 0\} \cup \{|s| < \delta\}.$$



We write the Taylor expansion of $L_f(s)$ at $s=1$:

$$L_f(s) = \sum_{k=0}^{\infty} \frac{L_f^{(k)}(1)}{k!}(s-1)^k \qquad (*)$$

One can check $L_f^{(k)}(1) = \sum_{n=1}^{\infty} \frac{f(n)}{n}(-\log n)^k$.

Also, the radius of convergence of the power series $(*)$ is the distance from 1 to nearest singularity of $L_f(s)$. Since $L_f(s)$ analytic in D, nearest points to 1 not in D are $\pm i\delta$, hence radius of convergence $\geq \sqrt{1+\delta^2} = 1+\delta'$

Let $-\delta' < -\sigma < 0$. Then

$$L_f(-\sigma) = \sum_{k=0}^{\infty} \frac{(-\sigma-1)^k}{k!} \sum_{n=1}^{\infty} \frac{(-\log n)^k \cdot f(n)}{n}$$

$$= \sum_{k=0}^{\infty} \frac{(1+\sigma)^k}{k!} \sum_{n=1}^{\infty} \frac{(\log n)^k \cdot f(n)}{n}$$

(note this is convergent series of positive numbers, so it can be rearranged)

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} \sum_{k=0}^{\infty} \frac{(1+\sigma)^k (\log n)^k}{k!}$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} \exp((1+\sigma)\log n) = \sum_{n=1}^{\infty} f(n) n^{\sigma}$$

But this is divergent, contradiction. $\square$

Strengthenings of Dirichlet theorem

Theorem Let χ be a non-principal character mod q.
Then:

(i) $\displaystyle\sum_{n \le x} \frac{\chi(n)\log n}{n} = -L'(1,\chi) + O_\chi\left(\frac{\log x}{x}\right)$.

(ii) $\displaystyle\sum_{n \le x} \frac{\chi(n)\Lambda(n)}{n} \ll_\chi 1$

(iii) $\displaystyle\sum_{p \le x} \frac{\chi(p)\log p}{p} \ll_\chi 1$.

(iv) $\displaystyle\sum_{p \le x} \frac{\chi(p)}{p} = \log L(1,\chi) - \sum_{p}\sum_{l \ge 2} \frac{\chi(p^l)}{l p^l} + O_\chi\left(\frac{1}{\log x}\right)$

(v) $\displaystyle\prod_{p \le x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} = L(1,\chi) + O_\chi\left(\frac{1}{\log x}\right)$.

Proof (i) Note that since $L(s,\chi)$ holomorphic for $Re(s) > 0$,

$$L'(s,\chi) = -\sum_{n} \frac{\chi(n)\log n}{n^s} \text{ for } Re(s) > 0.$$

(can take derivatives for holomorphic functions).

Here $$\sum_{n\le X} \frac{\chi(n)\log n}{n} = -L'(1,\chi) + \underbrace{\sum_{n>X}\frac{\chi(n)\log n}{n}}_{E(X).}$$

Denote $$S(X)=\sum_{n\le X}\chi(n)$$

Then $|S(X)|\le q \ll_\chi 1$.

Hence $$E(X) = \lim_{T\to\infty}\sum_{X<n\le T}\frac{\chi(n)\log n}{n}$$

$$=\lim_{T\to\infty}\left(\underbrace{\frac{S(T)\log T}{T}}_{\xrightarrow[T\to\infty]{} 0} - \frac{S(X)\log X}{X} + \underbrace{\int_X^T S(t)\left(\frac{\log t}{t}\right)' dt}_{\ll \int_X^T \left|\left(\frac{\log t}{t}\right)'\right| dt \xrightarrow[T\to\infty]{} \frac{\log X}{X}.}\right)$$

Hence $E(X)\ll_\chi \frac{\log X}{X}$. ✓

(ii) Since $\log = \Lambda * \mathcal{E}$, we have

$$\frac{\chi(n)\log n}{n} = \sum_{d\mid n} \underbrace{\frac{\Lambda(d)\chi(d)}{d}}_{g(d)}\cdot\underbrace{\frac{\chi(\frac{n}{d})}{n/d}}_{h(\frac{n}{d})} = g*h(n)$$

By convolution method,

$$\sum_{n\le X}\frac{\chi(n)\log n}{n} = \sum_{d\le X}\frac{\Lambda(d)\chi(d)}{d}\sum_{m\le \frac{X}{d}}\frac{\chi(m)}{m}.$$

But $$\sum_{m\le y}\frac{\chi(m)}{m} = L(1,\chi) - \sum_{m>y}\frac{\chi(m)}{m}$$

$$\sum_{m>y} \frac{\chi(m)}{m} = \lim_{T\to\infty} \sum_{y<m\le T} \frac{\chi(m)}{m}$$

$$= \lim_{T\to\infty} \Big(\cancel{\frac{S(T)}{T}} - \frac{S(y)}{y} + \underbrace{\int_y^T \frac{S(t)}{t^2}\,dt}_{\ll_\chi \frac{1}{y}} \Big) \ll_\chi \frac{1}{y}.$$

Hence $\displaystyle\sum_{n\le x} \frac{\chi(n)\log n}{n} = \sum_{d\le x} \frac{\Lambda(d)\chi(d)}{d}\left(L(1,\chi) + O_\chi\left(\frac{d}{x}\right)\right)$

$$= L(1,\chi) \sum_{d\le x} \frac{\Lambda(d)\chi(d)}{d} + O_\chi\Big(\frac{1}{x}\sum_{d\le x} \Lambda(d)\chi(d)\Big).$$

$$\Big|\sum_{d\le x}\Lambda(d)\chi(d)\Big| \le \sum_{d\le x}\Lambda(d) = \psi(x) \asymp x$$

(by Chebyshev)

$$\Rightarrow -L'(1,\chi) + O_\chi\left(\frac{\log x}{x}\right) = L(1,\chi)\sum_{d\le x}\frac{\Lambda(d)\chi(d)}{d} + O_\chi(1).$$

Conclusion follows since $L(1,\chi) \neq 0$.

(iii), (iv), (v) exercise (Problem sheet 7)

<u>Corollary</u>: If $(a,q)=1$, $x\ge 2$

(a) $\displaystyle\sum_{\substack{n\le x\\ n\equiv a(q)}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(q)}\log x + O_q(1)$;

(ii) $$\sum_{\substack{p \le x \\ p \equiv a \pmod q}} \frac{\log p}{p} = \frac{1}{\varphi(q)} \log x + O_q(1);$$

(iii) There exists a constant $b(q,a)$ s.t.

$$\sum_{\substack{p \le x \\ p \equiv a (q)}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + b(q,a) + O_q\left(\frac{1}{\log x}\right)$$

think about these as Mertens theorems in arithmetic progressions.

Perron formula

→ one of our most important tools to relate arithmetic properties of $f \in \mathcal{A}$ to $L_f(s)$.

Theorem (Perron)
Let $f \in \mathcal{A}$, $c > \sigma_a(f)$, $c > 0$ and $X > 0$. Then

$$\sum_{n \leq X}{}' f(n) = \frac{1}{2\pi i} \int_{(c)} L_f(s) X^s \frac{ds}{s},$$

$(c) = \{s \in \mathbb{C} : Re(s) = c\}$.

where

$$\sum_{n \leq X}{}' f(n) = \begin{cases} \sum_{n \leq X} f(n), & \text{if } X \notin \mathbb{Z} \\ \sum_{n < X} f(n) + \frac{f(X)}{2}, & \text{otherwise} \end{cases}$$

Remark: Here we define

$$\int_{(c)} L_f(s) X^s \frac{ds}{s} := \lim_{T \to \infty} \int_{c-iT}^{c+iT} L_f(s) X^s \frac{ds}{s}.$$

! Not true in general integral converges absolutely, need to be careful !.

At the heart of the theorem is the following lemma:

Central lemma: Let $\delta(y) = \begin{cases} 1, & \text{if } y > 1 \\ 1/2, & y = 1 \\ 0, & \text{if } 0 < y < 1. \end{cases}$

For all $c > 0$ and $T \geq 2$, we have

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{y^s \, ds}{s} = \delta(y) + O\left(\frac{y^c}{T|\log y|}\right) \qquad y \neq 1.$$

and for $y = 1$: $\displaystyle \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} ds = \frac{1}{2} + O\left(\frac{c}{T}\right).$

In particular, $\displaystyle \frac{1}{2\pi i} \int_{(c)} y^s \frac{ds}{s} = \lim_{T \to \infty} \int_{c-iT}^{c+iT} y^s \frac{ds}{s} = \delta(y).$

Heuristic proof of Perron:

$$\frac{1}{2\pi i} \int_{(c)} L_f(s) X^s \frac{ds}{s} = \frac{1}{2\pi i} \int_{(c)} \sum_{n=1}^{\infty} f(n) \left(\frac{X}{n}\right)^s \frac{ds}{s} =$$

$$= \sum_{n=1}^{\infty} f(n) \frac{1}{2\pi i} \int_{(c)} \left(\frac{X}{n}\right)^s \frac{ds}{s} \quad \text{(using Fubini)}$$

$$= \sum_{n=1}^{\infty} f(n) \delta\left(\frac{X}{n}\right) = \sum_{n \leq X}{}' f(n).$$

Proof of central lemma: Define $I_c(y,T) := \frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s \frac{ds}{s}$.

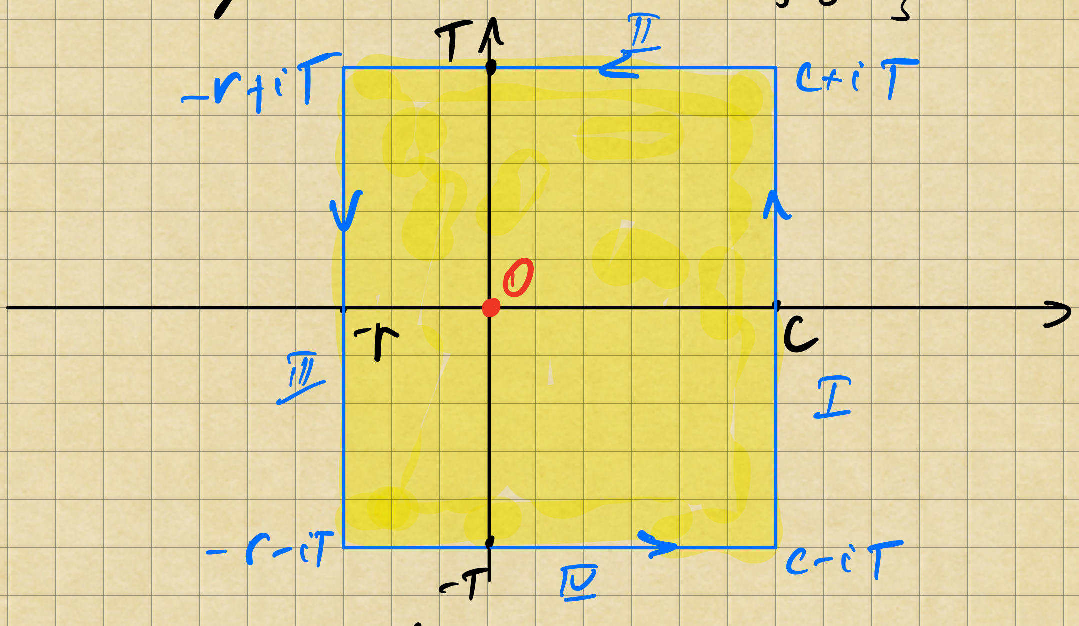
If $y=1$: $I_c(1,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{1}{s}\, ds =$

$$= \frac{1}{2\pi}\int_{-T}^{T} \frac{1}{c+it}\, dt = \frac{1}{2\pi}\int_{-T}^{T} \frac{c-it}{c^2+t^2}\, dt$$

(real part symmetric, imaginary part cancels) $= \frac{c}{\pi}\int_0^T \frac{1}{c^2+t^2}\, dt$

$$= \frac{1}{\pi}\int_0^{T/c} \frac{dt}{1+t^2} = \underbrace{\frac{1}{\pi}\int_0^{\infty} \frac{dt}{1+t^2}}_{=\frac{1}{2}} - \underbrace{\frac{1}{\pi}\int_{T/c}^{\infty} \frac{dt}{1+t^2}}_{O\left(\frac{c}{T}\right)}$$

Case $y>1$: Application of Cauchy's residue theorem.
Note that $\frac{y^s}{s}$ meromorphic in $s \in \mathbb{C}$, with simple pole at $s=0$ and $\operatorname{Res}_{s=0} \frac{y^s}{s} = 1$.



We integrate along the box with vertices $c-iT$, $c+iT$, $-r+iT$, $-r-iT$, for some $r>0$.

$$\underbrace{\int_{c-iT}^{c+iT} y^s \frac{ds}{s}}_{\mathrm{I}} + \underbrace{\int_{c+iT}^{-r+iT} y^s \frac{ds}{s}}_{\mathrm{II}} + \underbrace{\int_{-r+iT}^{-r-iT} y^s \frac{ds}{s}}_{\mathrm{III}} + \underbrace{\int_{-r-iT}^{c-iT} y^s \frac{ds}{s}}_{\mathrm{IV}} = 2\pi i \operatorname*{Res}_{s=0} \frac{y^s}{s} = 2\pi i .$$

II and IV: we have $|s| \geq T$, $|y^s| \leq y^{\mathrm{Re}(s)}$, so

$$\left| \int_{c-iT}^{-r+iT} y^s \frac{ds}{s} \right| \leq \int_{-r}^{c} \frac{y^\sigma}{T} \, d\sigma \leq \frac{1}{T} \int_{-\infty}^{c} y^\sigma \, d\sigma = \frac{y^c}{T|\log y|} .$$

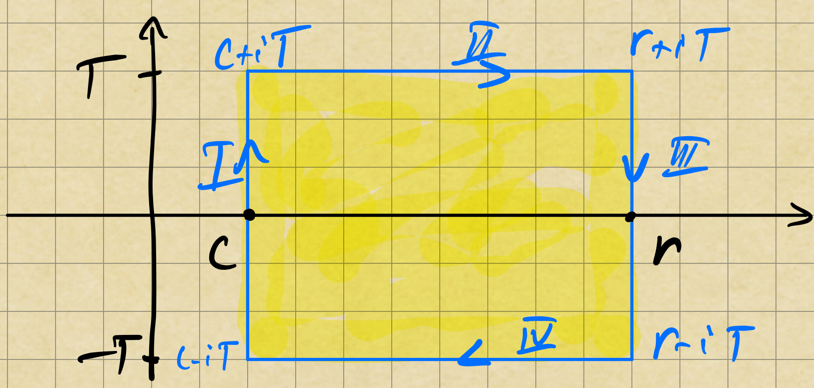
(similarly for IV).

For III: $|y^s| \leq y^{-r}$, $|s| \geq r$, hence

$$\left| \int_{-r+iT}^{-r-iT} y^s \frac{ds}{s} \right| \leq \int_{-T}^{T} \frac{y^{-r}}{r} \, dt \leq \frac{2y^{-r}T}{r} \xrightarrow{r\to\infty} 0$$

We let $r \to \infty$, this gives the result.

If $0<y<1$: We apply same argument for box $c-iT$, $c+iT$, $r+iT$, $r-iT$, where $r>1$.



$\frac{y^s}{s}$ is holomorphic in this region, horizontal integrals have same bound as previous case, and

$$\left| \int_{r+iT}^{r-iT} y^s \frac{ds}{s} \right| \leq \int_{-T}^{T} \frac{y^r}{r} \, dt = \frac{2y^r T}{r} \longrightarrow 0 \text{ as } r \to \infty. \qquad \boxtimes$$

We are now ready to provide a more precise version of Perron formula, which has many important applications.

Theorem (Truncated Perron formula)

Let $c > 0$, $T, X > 1$ and $f \in \mathcal{A}$.

If $c > \sigma_a(f)$, then

$$\sum_{n \leq X}{}' f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} L_f(s) X^s \frac{ds}{s} + E_{X,c}(T),$$

where $|E_{X,c}(T)| \ll \frac{X^c}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^c} + \max_{n \leq X} |f(n)| \left(1 + \frac{X \log X}{T}\right)$

Corollary: If $f(n) \ll (\log n)^K$ and $2 \leq T \leq 2X$,

Set $c = 1 + \frac{1}{\log X}$. Then

$$\sum_{n \leq X} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} L_f(s) X^s \frac{ds}{s} + O\left(\frac{X (\log X)^{K+1}}{T} \right).$$

Note that if $c = 1 + \frac{1}{\log X}$, $X^c = e \cdot X \ll X$,

$$\sum_{n=2}^{\infty} \frac{|f(n)|}{n^c} \ll \sum_{n=1}^{\infty} \frac{n^{\frac{c-1}{2}}}{n^c} = \zeta\left(\frac{1+c}{2}\right) = \frac{1}{\frac{1+c}{2}-1} + O(1)$$
$$\ll \log X.$$

<u>Applications:</u>

- counting primes

Let $c = 1 + \frac{1}{\log X}$ and $2 \le T \le 2X$. Then

$$\psi(X) = \sum_{n \le X} \Lambda(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} X^s \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{ds}{s} + O\left(\frac{X(\log X)^2}{T}\right)$$

- counting square-free numbers

$c = 1 + \frac{1}{\log X}$, $2 \le T \le 2X$

$$\sum_{n \le X} \mu^2(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s)}{\zeta(2s)} X^s \frac{ds}{s} + O\left(\frac{X \log X}{T}\right)$$

- generalised divisor function:

$$\tau_k(n) = \sum_{d_1 \dots d_k = n} 1 = \underbrace{\xi * \dots * \xi}_{k \text{ times}}(n)$$

We know $\tau_k(n) \ll_\varepsilon n^\varepsilon$, $\forall \varepsilon > 0$.

For $c = 1 + \frac{1}{\lg x}$, $\quad 2 \le T \le 2x$:

$$\sum_{n \le x} \tau_k(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \zeta^k(s) x^s \frac{ds}{s} + O\left(\frac{x^{1+\varepsilon}}{T}\right)$$

(check error term!)